Partial Differential Equations: Resit Exam

Room 5118.-152, Thursday 23 June 2016, 09:00 - 12:00

Duration: 3 hours

- Solutions should be complete and clearly present your reasoning.
- 10 points are "free". There are 6 questions and the total number of points is 100. The final exam grade is the total number of points divided by 10.
- Do not forget to very clearly write your **full name** and **student number** on the envelope.
- Do not seal the envelope.

Question 1 (14 points)

Consider the equation

$$2y\, u_x + e^x\, u_y = 0, \tag{1}$$

where $u = u(x, y)$.

(a) (10 points) Find the general solution of Eq. (1).

Solution

We consider the equation for the characteristic curves

$$\frac{dx}{dy} = \frac{2y}{e^x},$$

which can be separated as

$$e^x\, dx = 2y\, dy,$$

and directly integrated to

$$e^x = y^2 + C.$$

Therefore $C = e^x - y^2$ is the constant of integration and the general solution is

$$u = f(e^x - y^2).$$

(b) (4 points) Find the solution of Eq. (1) with the auxiliary condition $u(0, y) = y^2$.

Solution

For $u(0, y) = y^2$ we have $u(0, y) = f(1 - y^2) = y^2$. Therefore,

$$f(x) = 1 - x,$$

and this implies

$$u(x, y) = 1 - e^x + y^2.$$

Question 2 (14 points)

Consider the equation

$$u_{xx} + 2u_{xy} - 3u_{yy} = 0. \tag{2}$$

(a) (4 points) What is the type (elliptic / hyperbolic / parabolic) of Eq. (2)? Explain your answer.

Solution

We have $a_{11} = 1$, $a_{12} = 1$, and $a_{22} = -3$. Therefore

$$a_{11}a_{22} = -3 < 1 = a_{12}^2.$$

Therefore the equation is *hyperbolic*.

(b) (10 points) Find a linear transformation $(x, y) \to (s, t)$ that reduces Eq. (2) to one of the standard forms $u_{ss} + u_{tt} = 0$, $u_{ss} - u_{tt} = 0$, or $u_{ss} = 0$. Express the "old" coordinates (x, y) in term of the "new" coordinates (s, t). Which one of the standard forms is the correct one?

Solution

We complete the square in

$$\mathcal{L} = \partial_x^2 + 2\partial_x\partial_y - 3\partial_y^2 = (\partial_x + \partial_y)^2 - (2\partial_y)^2.$$

Define

$$\partial_s = \partial_x + \partial_y, \quad \partial_t = 2\partial_y.$$

Then, using the transpose of the linear mapping, we have

$$x = s, \quad y = s + 2t,$$

and the corresponding standard form is $u_{ss} - u_{tt} = 0$.

Question 3 (16 points)

Consider the eigenvalue problem $-X''(x) = \lambda X(x)$, $0 \le x \le 2\pi$, with boundary conditions $X(0) = 0$ and $X'(2\pi) = 0$.

(a) (6 points) Show that the given boundary conditions are of the form

$$\alpha_1 X(a) + \beta_1 X(b) + \gamma_1 X'(a) + \delta_1 X'(b) = 0,$$
$$\alpha_2 X(a) + \beta_2 X(b) + \gamma_2 X'(a) + \delta_2 X'(b) = 0,$$

and that if a function $f(x)$ satisfies the given boundary conditions then

$$f(x)f'(x)|_0^{2\pi} = 0.$$

What can you conclude from these facts about the eigenvalues in this problem?

Solution

To show that the given periodic boundary conditions are of the form

$$\begin{aligned} \alpha_1 X(a) + \beta_1 X(b) + \gamma_1 X'(a) + \delta_1 X'(b) = 0, \\ \alpha_2 X(a) + \beta_2 X(b) + \gamma_2 X'(a) + \delta_2 X'(b) = 0, \end{aligned} \tag{3}$$

take $a = 0$, $b = 2\pi$, $\gamma_1 = \delta_1 = \alpha_2 = \beta_2 = \beta_1 = \gamma_2 = 0$, $\alpha_1 = \delta_2 = 1$.

We have that

$$f(x)f'(x)|_0^{2\pi} = f(2\pi)f'(2\pi) - f(0)f'(0) = f(2\pi) \cdot 0 - 0 \cdot f'(0) = 0.$$

We know that if the periodic boundary condition are in the form of Eq. (3) and if $f(x)f'(x)|_0^{2\pi} \le 0$ then there are no negative eigenvalues.

(b) (10 points) It is given that all eigenvalues are real. Prove that they are given by $\lambda_n = (2n+1)^2/16$, $n = 0, 1, 2, \ldots$ and give the corresponding eigenfunctions. Check whether $\lambda = 0$ is an eigenvalue.

Solution

Since the eigenvalues are real and we know that there are no negative eigenvalues we have to consider the cases $\lambda = \beta^2 > 0$ and $\lambda = 0$.

For $\lambda = 0$ we have $X'' = 0$ which gives the solution

$$X_0(x) = Ax + B.$$

Since $X_0(0) = 0$ we find that $B = 0$. The equation $X_0'(2\pi) = 0$ gives $A = 0$. Therefore, $X_0(x) = 0$ which must be excluded.

For $\lambda = \beta^2 > 0$ we get the solution

$$X(x) = A\sin(\beta x) + B\cos(\beta x).$$

The boundary condition $X(0) = 0$ gives

$$B = 0.$$

The boundary condition $X'(2\pi) = 0$ gives

$$\beta A \cos(2\pi\beta) = 0.$$

Since A must be non-zero and since $\beta > 0$ then we have $\cos(2\pi\beta) = 0$ which gives

$$2\pi\beta_n = n\pi + \frac{\pi}{2} = \frac{2n+1}{2}\pi, \quad n = 0, 1, 2, \ldots,$$

so

$$\beta_n = \frac{2n+1}{4}, \quad n = 0, 1, 2, \ldots.$$

The eigenvalues are

$$\lambda_n = \frac{(2n+1)^2}{16}, \quad n = 0, 1, 2, \ldots.$$

The corresponding eigenfunctions have the form

$$\sin(\beta_n x) = \sin\left(\frac{2n+1}{4}x\right).$$

Question 4 (14 points)

Consider the Laplace equation

$$u_{xx} + u_{yy} = 0, \tag{4}$$

in the domain $0 < x < \pi$, $0 < y < 1$.

(a) (4 points) Separate variables using a solution of the form $u(x, y) = X(x)Y(y)$ and find the ordinary differential equations satisfied by $X(x)$ and by $Y(y)$.

Solution

We have

$$X''Y + XY'' = 0,$$

therefore

$$\frac{X''}{X} = -\frac{Y''}{Y} = -\lambda.$$

The two equations are

$$-X'' = \lambda X, \qquad Y'' = \lambda Y.$$

(b) (10 points) Find the solution $u(x, y)$ of Eq. (4) which satisfies the boundary conditions $u(x, 0) = \sum_{n=1}^{\infty} c_n \sin(nx)$ and $u(x, 1) = u(0, y) = u(2\pi, y) = 0$.

It is given that the eigenvalues for the problem $-Z'' = \mu Z$ with $Z(0) = Z(c) = 0$ are $\mu_n = n^2\pi^2/c^2$, for $n = 1, 2, 3, \ldots$, and the corresponding eigenfunctions are $\psi_n = \sin(n\pi x/c)$.

Solution

We have

$$-X'' = \lambda X, \quad X(0) = X(a) = 0,$$

and therefore the eigenvalues are

$$\lambda_n = n^2, \quad n = 1, 2, 3, \ldots.$$

The corresponding eigenfunctions are

$$\sin(nx).$$

Moreover, we have

$$Y'' = \lambda_n Y, \quad Y(b) = 0.$$

This means

$$Y_n = A_n \cosh(ny) + B_n \sinh(ny),$$

and

$$A_n \cosh(n) + B_n \sinh(n) = 0,$$

implying

$$A_n = -B_n \tanh(n).$$

Finally

$$Y_n = B_n(-\tanh(n)\cosh(ny) + \sinh(ny)).$$

The general solution is

$$u(x,y) = \sum_{n=1}^{\infty} B_n \sin(nx)(-\tanh(n)\cosh(ny) + \sinh(ny)).$$

For $y = 0$ we get

$$u(x,0) = \sum_{n=1}^{\infty} -B_n \tanh(n)\sin(nx) = \sum_{n=1}^{\infty} c_n \sin(nx),$$

thus,

$$B_n = -c_n/\tanh(n).$$

The final solution is

$$u(x,y) = \sum_{n=1}^{\infty} c_n \sin(nx)\frac{\sinh(n)\cosh(ny) - \cosh(n)\sinh(ny)}{\sinh(n)}.$$

Question 5 (16 points)

Consider the function

$$f(x) = \begin{cases} 1, & \text{for } 0 \le x < 1/2, \\ -1, & \text{for } 1/2 \le x \le 1, \end{cases}$$

and its Fourier cosine series

$$\frac{b_0}{2} + \sum_{n=1}^{\infty} b_n \cos(n\pi x).$$

(a) (4 points) Check if the Fourier series converges to $f(x)$ in the L^2 sense in the interval $[0,1]$.

Solution

The Fourier series converges because the integral

$$\|f\|^2 = \int_0^1 |f(x)|^2\, dx$$

is finite.

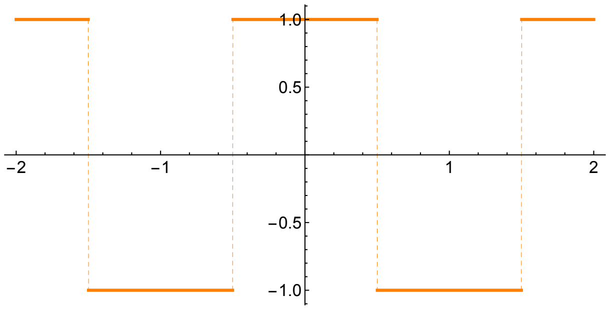
(b) (6 points) What is the pointwise limit of the Fourier series for $x \in [-2,2]$?

Solution

To answer this question we must look at the even-periodic extension $f_{\text{ext}}(x)$ of $f(x)$. The even extension is,

$$f_{\text{even}}(x) = \begin{cases} -1, & \text{for } -1 \le x \le -1/2, \\ 1, & \text{for } -1/2 < x < 1/2, \\ -1, & \text{for } 1/2 \le x \le 1. \end{cases}$$

The periodic extension is obtained by repeating the graph of $f_{\text{even}}(x)$, $x \in [-1, 1]$ with period 2, as in the graph below.

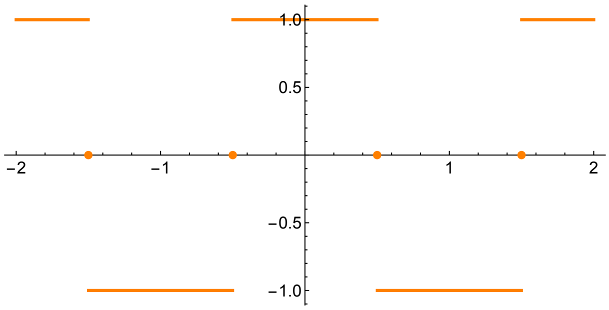


Therefore the even-periodic extension $f_{\text{ext}}(x)$ is discontinuous at $k + 1/2$, $k \in \mathbb{Z}$. For $x \in [-2, 2]$ we have discontinuities at $x \in D$ where $D = \{-1.5, -0.5, 0.5, 1.5\}$.

Therefore the pointwise limit of the Fourier series equals $f_{\text{ext}}(x)$ for $x \in [-2, 2] \setminus D$ and it equals $[f_{\text{ext}}(x^+) + f_{\text{ext}}(x^-)]/2$ for $x \in D$. At each $x \in D$ the two side-limits are ± 1 so $[f_{\text{ext}}(-1^+) + f_{\text{ext}}(-1^-)]/2 = 0$. Therefore at each $x \in D$ the pointwise limit of the Fourier series is 0.

(c) (2 points) Draw the graph of the Fourier series for $x \in [-2, 2]$.

Solution



(d) (4 points) At which points in $[0, 1]$ does the Gibbs phenomenon appear in the Fourier series and what is the overshoot at these points?

Solution

The Gibbs phenomenon appears at the points in $[0, 1]$ where $f_{\text{ext}}(x)$ is discontinuous, that is, at $x = 1/2$. The discontinuity jump is $f_{\text{ext}}(1^+) - f_{\text{ext}}(1^-) = -2$, which means that the overshoot is $\simeq 0.09 \times 2 = 0.18$.

Question 6 (16 points)

(a) (4 points) Suppose that u is a harmonic function in the disk $D = \{r < 1\}$ and that for $r = 1$ we have $u(1, \theta) = 1 + 3\cos\theta + 2\sin 3\theta$. Find the solution $u(r, \theta)$ for $r \leq 1$.

It is given that the solution to the Laplace equation inside the disk $r < a$ has the form

$$u(r, \theta) = \frac{C_0}{2} + \sum_{n=1}^{\infty} \frac{r^n}{a^n} \left(C_n \cos(n\theta) + D_n \sin(n\theta) \right).$$

Solution

Setting $r = 1$ in the expression for $u(r, \theta)$ (where $a = 1$) we find

$$u(1, \theta) = \frac{C_0}{2} + \sum_{n=1}^{\infty} C_n \cos(n\theta) + D_n \sin(n\theta),$$

Comparing with the given expression for $u(1,\theta)$ we get $C_0 = 2,\ C_1 = 3,\ D_3 = 2,$ and all other coefficients are zero. Therefore the solution is

$$u(r,\theta) = \frac{C_0}{2} + C_1 r\cos\theta + D_3 r^3 \sin 3\theta = 1 + 3r\cos\theta + 2r^3\sin 3\theta.$$

(b) (12 points) Consider a solution $w(x,t)$ of the wave equation $w_{tt} = w_{xx}$ for x in $[0,\ell]$ and homogeneous Neumann boundary conditions, $w_x(0,t) = w_x(\ell,t) = 0$. Show that the energy,

$$E(t) = \tfrac{1}{2}\int_0^\ell (w_t^2 + w_x^2)dx,$$

is constant. If $w(x,0) = 3\ell x^2 - 2x^3$ and $w_t(x,0) = 0$ show that $E(t) = (6/10)\ell^5$ for all $t \geq 0$.

Solution

We have that

$$\frac{dE}{dt} = \int_0^\ell (w_t w_{tt} + w_x w_{xt})\, dx = \int_0^\ell (w_t w_{xx} + w_x w_{xt})\, dx = \int_0^\ell (w_t w_x)_x\, dx = w_t w_x|_0^\ell = 0.$$

For $t = 0$ we have

$$E(0) = \frac{1}{2}\int_0^\ell (w_t^2 + w_x^2)\, dx = \frac{1}{2}\int_0^\ell w_x^2\, dx,$$

since $w_t = 0$. Moreover,

$$w_x = 6(\ell x - x^2),$$

so

$$E(0) = 18\int_0^\ell (\ell^2 x^2 + x^4 - 2\ell x^3)\, dx = 18\left(\frac{\ell^5}{3} + \frac{\ell^5}{5} - \frac{\ell^5}{2}\right) = \frac{6\ell^5}{10}.$$

Since $dE/dt = 0$ we conclude $E(t) = E(0) = (6/10)\ell^5$ for all $t \geq 0$.

End of the exam (Total: 90 points)